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# Liouvillian first integrals of quadratic–linear polynomial differential systems

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ABSTRACT

For a large class of quadratic–linear polynomial differential systems with a unique singular point at the origin having non-zero eigenvalues, we classify the ones which have a Liouvillian first integral, and we provide the explicit expression of them.



## 1. Introduction

For planar differential systems the notion of integrability is based on the existence of a first integral. For such systems the existence of a first integral determines completely its phase portrait. Then a natural question arises: *Given a system of ordinary differential equations in* $\mathbb{R}^2$ *depending on parameters, how to recognize the values of such parameters for which the system has a first integral?*

In particular the planar integrable systems which are not Hamiltonian, i.e. the systems in $\mathbb{R}^2$ that cannot be written as $x' = -\partial H/\partial y$, $y' = \partial H/\partial x$ for some function $H : \mathbb{R}^2 \to \mathbb{R}$ of class $C^2$, are in general very difficult to detect. Here the prime denotes derivative with respect to the independent variable $t$.

The first step to detect those first integrals in different classes of functions, namely polynomial, rational, elementary or Liouvillian, is to determine the algebraic invariant curves (i.e., the so-called Darboux polynomials).

Let $P$ and $Q$ be two real polynomials in the variables $x$ and $y$, then the system

$$x' = P(x, y), \qquad y' = Q(x, y), \tag{1}$$

is a *quadratic polynomial differential system* if the maximum of the degrees of the polynomials $P$ and $Q$ is two.

Quadratic polynomial differential systems have been investigated for many authors, and more than one thousand papers have been published about these systems (see for instance [14] and [16]), but the problem of classifying all the integrable quadratic polynomial differential systems remains open.

Let $U \subset \mathbb{R}^2$ be an open set. We say that the non-constant function $H : U \to \mathbb{R}$ is a first integral of the polynomial vector field $X$ on $U$, if $H(x(t), y(t)) =$ constant for all values of $t$ for which the solution $(x(t), y(t))$ of $X$ is defined on $U$. Clearly $H$ is a first integral of $X$ on $U$ if and only if $XH = 0$ on $U$.

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The study of the Liouvillian first integrals is a classical problem of the integrability theory of the differential equations which goes back to Liouville, see for details again [15]. A *Liouvillian first integral* is a first integral $H$ which is a Liouvillian function, that is, roughly speaking which can be obtained "by quadratures" of elementary functions. For a precise definition see [15].

As far as we know the Liouvillian first integral of some multi-parameter family of planar polynomial differential systems has only been completed classified for the planar Lotka–Volterra system of degree 2, see [3,10–13].

It was proved in [9] (see Proposition 3), that any quadratic–linear differential system

$$x' = a_{00} + a_{10}x + a_{01}y + a_{20}x^2 + a_{11}xy + a_{02}y^2, \qquad y' = A + Bx + Cy, \tag{2}$$

with $A^2 + B^2 + C^2 \neq 0$ having a unique finite singular point with non-zero eigenvalues, through a linear change of variables and a rescaling of the time can be written into the form

$$x' = P(x, y) = bx + cy + dx^2 + exy + fy^2, \qquad y' = Q(x, y), \tag{3}$$

where $P(x, y) \neq bx + cy$ and $Q(x, y)$ is either $x$ or $y$. Moreover,

(S1) if $Q(x, y) = y$, then $d = 0$, $b \neq 0$ and $e^2 + f^2 \neq 0$;
(S2) if $Q(x, y) = x$, then $f = 0$, $c \neq 0$ and $d^2 + e^2 \neq 0$.

We do not consider in (3) the case $Q(x, y) = 0$ because the possible singular points have always a Jacobian matrix with zero eigenvalues. We do not consider in (3) the case $Q(x, y) = 1$ since it has no singular points. When $Q(x, y) = y$ then $d = 0$ and $b \neq 0$. Indeed, the singular points of (3) satisfy in this case $y = 0$ and $P(x, y) = x(b + dx) = 0$. Therefore, since we want the origin to be the unique singular point we must have $bd = 0$ with $b^2 + d^2 \neq 0$. If $d \neq 0$ then $b = 0$ and in this case the Jacobian matrix at the origin has a zero eigenvalue, so we do not consider this case. Therefore we must have $d = 0$ and $b \neq 0$. Furthermore, since $P(x, y)$ must be quadratic and $d = 0$, we must have $e^2 + f^2 \neq 0$. Proceeding in a similar way when $Q(x, y) = x$ in order that it has only the origin as a singular point with Jacobian having non-zero eigenvalues, we must have $f = 0$ and $c \neq 0$. Furthermore, in order that $P(x, y)$ be quadratic we must have $d^2 + e^2 \neq 0$.

Our first result is the following.

**Theorem 1.** *System* (3) *satisfying* (S1) *is integrable.*

(a) *If $e \neq 0$, then the first integral is*

$$H = \exp(-ey)y^{-b}\big(ex + fy + \exp(ey)\big(ce + (1 - b)f\big)yEI_b(ey)\big), \tag{4}$$

*where $EI_b(x)$ is the exponential integral function*

$$EI_b(x) = \int_1^{\infty} \frac{\exp(-xt)}{t^b}\,dt = x^{b-1}\Gamma(1 - b, x) \quad \textit{for any } b \in \mathbb{R},$$

*where $\Gamma$ is the incomplete gamma function, for more details see* [1].

(b) *If $e = 0$ and $(b - 1)(b - 2) \neq 0$, then the first integral is*

$$H = y^{-b}\big((b - 1)(b - 2)x + y\big((b - 2)c + (b - 1)fy\big)\big). \tag{5}$$

(c) *If $e = 0$ and $b = 1$, then the first integral is*

$$H = \frac{x}{y} - fy - c\log y. \tag{6}$$

(d) *If $e = 0$ and $b = 2$, then the first integral is*

$$H = \frac{x + cy}{y^2} - f\log y. \tag{7}$$

The proof of Theorem 1 is given in Section 3.

Our second result is the following.

**Theorem 2.** *The unique Liouvillian first integrals $H = H(x, y)$ of system* (3) *satisfying* (S2) *are*:

(a) $H = (c + ex)^{c/e^2}\exp(y^2/2 - x/e)$ *if $d = b = 0$*;
(b) $H = \exp(-2dy)(2cdy + c + 2d^2x^2)$ *if $b = e = 0$*.

The proof of Theorem 2 is given in Section 6. For proving Theorem 2 we need to characterize the Darboux polynomials and the exponential factors of system (3) satisfying (S2). The analytic first integrals of system (3) satisfying either (S1), or (S2) were classified in [9].

Note that the results of this paper are an important step towards the almost impossible (by now) task of classifying all quadratic–linear polynomial differential systems having a Liouvillian first integral.

The quadratic–linear differential systems (2) studied in this paper appears in many different areas of the sciences, we shall select three of their applications. Thus, for instance Birkhoff and Smith [2] in their study of the structure of the surface transformations they need at some moment to consider the quadratic–linear differential system

$$\dot{x} = y, \qquad \dot{y} = -x - \alpha y - \mu x^2 - y^2,$$

with $\alpha, \mu > 0$.

The flow of a perfect gas with constant specific heats through a rotating channel of constant cross-sectional area, as used in certain helicopter propulsion systems and wind-driven gas turbines, is described by an equation of the form

$$\dot{x} = 1 + \delta y + \gamma x^2, \qquad \dot{y} = y(\alpha x + \beta y),$$

where the real parameter $\gamma$ may be zero, providing quadratic–linear differential systems, see the work of Kestin and Zaremba [6].

The Falkner–Skan equation $y''' + y''y + \lambda(1 - y'^2) = 0$ appears related to the study of boundary layer problems in fluid dynamics. A key point for understanding the origin of his period orbits is played by the quadratic–linear differential system

$$\dot{y} = z, \qquad \dot{z} = 1 - y^2.$$

See for more details the paper of Llibre and Messias [8].

## 2. Preliminary results

As it is explained in [7] the study of complex invariant algebraic curves and of complex exponential factors is necessary for obtaining all the real Liouvillian first integrals of the real polynomial differential equations.

The vector field $X$ associated to system (1) is defined by

$$X = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y}.$$

We say that $h = h(x, y) = 0$ with $h(x, y) \in \mathbb{C}[x, y] \setminus \mathbb{C}$ is an *invariant algebraic curve* of the vector field $X$ if it satisfies

$$Xh = P\frac{\partial h}{\partial x} + Q\frac{\partial h}{\partial y} = Kh.$$

The polynomial $K = K(x, y) \in \mathbb{C}[x, y]$ is called the *cofactor* of $h = 0$ and has degree at most $m - 1$. The polynomial $h$ is called a *Darboux polynomial*, and we also say that $K$ is the *cofactor* of the Darboux polynomial $h$.

It was proved in [4] that $h$ is a Darboux polynomial of system (1) with cofactor $K_h$ if and only if the factorization of $h$ into irreducible factors in $\mathbb{C}[x, y]$, i.e., $h = h_1^{n_1} \cdots h_r^{n_r}$, satisfies that each $h_i$ is a Darboux polynomial for $i = 1, \dots, r$ with cofactor $K_{h_i}$.

In [9] the authors proved that system (3) satisfying (S2) has no polynomial first integrals, that is, it has no Darboux polynomials with zero cofactor.

**Proposition 3.** *System* (3) *satisfying* (S2) *with* $b^2 + d^2 \neq 0$, $b^2 + e^2 \neq 0$ *and having an irreducible invariant algebraic curve* $h = h(x, y) = 0$ *is the following*:

(a) $h = x + c/e$, *if* $de \neq 0$ *and* $b = dc/e$, *with cofactor* $K = dx + ey$,
(b) $h = x + ey/d$ *if* $de \neq 0$ *and* $b = dc/e - e/d$, *with cofactor* $K = d(c + ex)/e$.

Proposition 3 will be proved in Section 4.

We say that $E = \exp(g/h)$, with $g, h \in \mathbb{C}[x, y]$, $(g, h)$ coprime and $E \notin \mathbb{C}$, is an *exponential factor* of the vector field $X$ if it satisfies

$$XE = P\frac{\partial E}{\partial x} + Q\frac{\partial E}{\partial y} = KE, \quad \text{i.e.} \quad P\frac{\partial(g/h)}{\partial x} + Q\frac{\partial(g/h)}{\partial y} = K.$$

The polynomial $K = K(x, y) \in \mathbb{C}[x, y]$ is called the *cofactor* of $E$ and has degree at most $m - 1$.

**Proposition 4.** *The following statements hold.*

(a) *If $E = \exp(g/h)$ is an exponential factor for the polynomial system* (1) *and $h$ is not a constant polynomial, then $h = 0$ is an invariant algebraic curve.*
(b) *Eventually* $\exp(g)$ *can be an exponential factor, coming from the multiplicity of the infinite invariant straight line.*

For a geometrical meaning of the exponential factors and a proof of Proposition 4 see [5].

**Proposition 5.** *System* (3) *satisfying* (S2) *with $b^2 + d^2 \neq 0$, $b^2 + e^2 \neq 0$ and having an exponential factor is the following*:

(a) $\exp(y)$ *and* $\exp(-x/b + y + ey^2/(2b))$ *if $d = 0$ and $b \neq 0$, with cofactors $x$ and $-cy/b$, respectively.*
(b) $\exp(y)$ *in any other case with cofactor $x$.*

The proof of Proposition 5 is given in Section 5.

A non-constant complex function $R : \mathbb{C}^2 \to \mathbb{C}$ is an *integrating factor* of the polynomial vector field $X$ on $U$, if one of the following three equivalent conditions holds

$$\frac{\partial(RP)}{\partial x} = -\frac{\partial(RQ)}{\partial y}, \qquad \operatorname{div}(RP, RQ) = 0, \qquad XR = -R\operatorname{div}(P, Q),$$

on $U$. As usual the *divergence* of the vector field $X$ is given by

$$\operatorname{div}(P, Q) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}.$$

The next result summarizes the main results about the Darboux theory of integrability that we shall use in this paper.

**Theorem 6.** *Suppose that the polynomial vector field $X$ of degree $m$ defined in $\mathbb{C}^2$ admits $p$ invariant algebraic curves $f_i = 0$ with cofactors $K_i$, for $i = 1, \ldots, p$ and $q$ exponential factors $E_j = \exp(g_j/h_j)$ with cofactors $L_j$, for $j = 1, \ldots, q$. Then the following statements hold.*

(a) *There exist $\lambda_i, \mu_j \in \mathbb{C}$ not all zero such that*

$$\sum_{i=1}^{p} \lambda_i K_i + \sum_{j=1}^{q} \mu_j L_j = 0 \tag{8}$$

*if and only if the function*

$$f_1^{\lambda_1} \cdots f_p^{\lambda_p} E_1^{\mu_1} \cdots E_q^{\mu_q} \tag{9}$$

*is a first integral of $X$ (such a function is called a* Darboux function*).*
(b) *There exist $\lambda_i, \mu_j \in \mathbb{C}$ not all zero such that*

$$\sum_{i=1}^{p} \lambda_i K_i + \sum_{j=1}^{q} \mu_j L_j = -\operatorname{div}(P, Q), \tag{10}$$

*if and only if the function of Darboux type* (9) *is an integrating factor of $X$.*
(c) *If $p + q = [m(m+1)/2] + 1$, then there exist $\lambda_i, \mu_j \in \mathbb{C}$ not all zero such that* (8) *holds. If $p + q = m(m+1)/2$, then there exist $\lambda_i, \mu_j \in \mathbb{C}$ not all zero such that either* (8), *or* (10) *holds.*

To prove the results related with Liouvillian first integrals we use the following result proved in [15].

**Theorem 7.** *The polynomial differential system* (1) *has a Liouvillian first integral if and only if it has an integrating factor which is a Darboux function.*

## 3. Proof of Theorem 1

System (3) satisfying (S1) can be written as the following linear differential system

$$\frac{dx}{dy} = \frac{b + ey}{y} x + c + fy. \tag{11}$$

We consider different cases.

*Case* 1: $e \neq 0$. In this case the general solution of (11) is

$$x(y) = \frac{-fy - e\exp(ey)y^b H - \exp(ey)(ce + (1-b)f)yEI_b(ey)}{e},$$

where $H$ is a first integral, since it is an integration constant. Hence system (3) satisfying (S1) is integrable with first integral $H$ as in (4).

*Case* 2: $e = 0$ *and* $(b-1)(b-2) \neq 0$. Then the general solution of (11) is

$$x(y) = \frac{cy}{1-b} + \frac{fy^2}{2-b} + y^b H,$$

where $H$ is an integration constant. Hence system (3) satisfying (S1) is integrable with first integral $H$ as in (5).

*Case* 3: $e = 0$ *and* $b = 1$. Then the general solution of (11) is

$$x(y) = y(fy + H + c\log y),$$

where $H$ is again an integration constant. Hence system (3) satisfying (S1) is integrable with first integral $H$ as in (6).

*Case* 4: $e = 0$ *and* $b = 2$. Then the general solution of (11) is

$$x(y) = y(-c + yH + fy\log y).$$

Hence system (3) satisfying (S1) is integrable with first integral $H$ as in (7). The proof of Theorem 1 is now completed.

## 4. Darboux polynomials of system (3) satisfying (S2)

System (3) satisfying (S2) with $d = 0$ and $b = 0$ is

$$x' = y(c + ex), \qquad y' = x,$$

which clearly is Liouvillian integrable with the first integral of Theorem 2(a). Moreover, system (3) satisfying (S2) with $b = e = 0$ is

$$x' = cy + dx^2, \qquad y' = x,$$

and it is easy to check that it has the Liouvillian first integral given in Theorem 2(b). Hence, *from now on we will assume that* $b^2 + d^2 \neq 0$, *and* $b^2 + e^2 \neq 0$.

In view of the results of [9] we have that system (3) satisfying (S2) has no Darboux polynomials with zero cofactor. Therefore we only need to study the Darboux polynomials with non-zero cofactor. Proposition 3 follows from the following three lemmas.

**Lemma 8.** *System* (3) *satisfying* (S2) *with* $d = 0$ *and* $b \neq 0$, *or* $d \neq 0$ *and* $e = 0$ *has no Darboux polynomials with non-zero cofactor.*

**Proof.** We consider two different cases.

*Case* 1: $d = 0$ *and* $b \neq 0$. System (3) is

$$x' = bx + cy + exy, \qquad y' = x, \tag{12}$$

with $bce \neq 0$. We will show that this system has no irreducible Darboux polynomial with non-zero cofactor. We proceed by contradiction. Let $h = h(x, y)$ be an irreducible Darboux polynomial with non-zero cofactor. Then it satisfies

$$(bx + cy + exy)\frac{\partial h}{\partial x} + x\frac{\partial h}{\partial y} = (\alpha_0 + \alpha_1 x + \alpha_2 y)h. \tag{13}$$

We write $h$ in sum of its homogeneous parts as $h = \sum_{j=0}^n h_j(x, y)$ where each $h_j(x, y)$ is a homogeneous polynomial in the variables $x$, $y$. Since $h$ is not a constant we must have $h_n \neq 0$ and $n > 0$. Moreover, $h_n$ satisfies

$$exy\frac{\partial h_n}{\partial x} = (\alpha_1 x + \alpha_2 y)h_n \quad \text{that is} \quad h_n = c_n(y)\exp\bigl(\alpha_1 x/(ey)\bigr)x^{\alpha_2/e}, \quad c_n(y) \neq 0.$$

Since $h_n$ must be a polynomial we get $\alpha_1 = 0$, $\alpha_2 = me$ with $0 \leqslant m \leqslant n$, and $h_n = c_n y^{n-m} x^m$ with $c_n \in \mathbb{C} \setminus \{0\}$.

Now computing the terms of degree $n$ in (13) we obtain

$$exy\frac{\partial h_{n-1}}{\partial x} + (bx + cy)c_n m x^{m-1} y^{n-m} + xc_n x^m (n-m) y^{n-m-1} = emyh_{n-1} + \alpha_0 c_n x^m y^{n-m}.$$

Hence

$$h_{n-1} = c_{n-1}x^m y^{n-m-1} + \frac{c_n cm}{e}x^{m-1}y^{n-m} + \frac{c_n(m-n)}{e}x^{m+1}y^{n-m-2} + \frac{c_n}{e}x^m y^{n-m-1}(\alpha_0 - bm)\log x.$$

Since $h_{n-1}$ must be a polynomial and $c_n \neq 0$ we get that $\alpha_0 = bm$ and

$$h_{n-1} = c_{n-1}x^m y^{n-m-1} + \frac{c_n cm}{e}x^{m-1}y^{n-m} + \frac{c_n(m-n)}{e}x^{m+1}y^{n-m-2}.$$

Furthermore, since $h_{n-1}$ is a polynomial and $c_n c \neq 0$ we must have $m \geqslant 1$ and $n \geqslant 1$. If $n = m+1$ since $c_n \neq 0$ we would have $m = n$ which is not possible. If $m = n$ then

$$c_{n-1} = 0 \quad \text{and} \quad h_{n-1} = \frac{c_n cn}{e}x^{n-1}.$$

Now computing the terms of degree $n-1$ in (13) we obtain

$$exy\frac{\partial h_{n-2}}{\partial x} + (bx+cy)\frac{\partial h_{n-1}}{\partial x} + x\frac{\partial h_{n-1}}{\partial y} = enyh_{n-2} + bnh_{n-1}.$$

Solving it we get

$$h_{n-2} = c_{n-2}x^n y^{-2} + \frac{c^2 c_n(n-1)n}{2e^2}x^{n-2} - \frac{bcc_n n}{e^2}x^{n-1}y^{-1}.$$

Since $h_{n-2}$ must be a polynomial and $bcc_n ne \neq 0$ we reach a contradiction. Hence $n \geqslant m+2$ and in this case computing the terms of degree $n-1$ in (13) we obtain

$$exy\frac{\partial h_{n-2}}{\partial x} + (bx+cy)\frac{\partial h_{n-1}}{\partial x} + x\frac{\partial h_{n-1}}{\partial y} = emyh_{n-2} + bmh_{n-1}.$$

Solving it we get

$$\begin{aligned} h_{n-2} = {} & \frac{x^{m-2}y^{-m+n-4}}{2e^2}\big[2c_n cx^2y^2(n-m)\log x + c_n(m-n)(2+m-n)x^4 \\ & + 2\big(c_{n-1}e(1+m-n) + bc_n(n-m)\big)x^3y + 2c(c_{n-1}e - bc_n)mxy^3 + c^2c_n(m-1)my^4\big]. \end{aligned}$$

Since $h_{n-2}$ must be a polynomial we must have $c_n c(n-m) = 0$, a contradiction with the fact that $c_n c \neq 0$ and $n \geqslant m+2$.

*Case* 2: $d \neq 0$ *and* $e = 0$. In this case system (3) satisfying (S2) becomes

$$x' = bx + cy + dx^2, \qquad y' = x.$$

We introduce the change of variables $(X, Y)$ with $x = X$ and $y = (Y - c - 2d^2x^2)/(2cd)$. Then we have that

$$X' = -\frac{c}{2d} + bX + \frac{Y}{2d}, \qquad Y' = 2dXY + 4bd^2X^2. \tag{14}$$

Doing the change of variables $(X, Y, t) \to (X, Z, s)$ with $Z = Y + 2bdX + c$ and $t = 2ds$, system (14) becomes

$$\dot{X} = Z, \qquad \dot{Z} = 4d^2(cX + bZ + XZ),$$

where the dot denotes derivative with respect to $s$. Now changing $(X, Z) \to (x, y)$ with $x = Z$ and $y = X$, we obtain system (12) for which we know from Case 1 that it has no Darboux polynomials with non-zero cofactor. So the lemma is proved. □

**Lemma 9.** *System* (3) *satisfying* (S2) *with* $de \neq 0$ *and* $b = dc/e + \gamma$ *with* $\gamma \in \{0, -e/d\}$ *has the following Darboux polynomials with non-zero cofactor*:

(a) $h = x + c/e$ *if* $\gamma = 0$ *with cofactor* $dx + ey$;
(b) $h = x + ey/d$ *if* $\gamma = -e/d$ *with cofactor* $d(c + ex)/e$.

**Proof.** We consider two different cases.

*Case* 1: $\gamma = 0$. It is easy to check that system (3) satisfying (S2) has $h_1 = x + c/e$ as the unique irreducible Darboux polynomial of degree one with non-zero cofactor $dx + ey$. We will see that it is the only one. We introduce the variables $(X, y)$ with $X = h_1$. In these variables system (3) satisfying (S2) becomes

$$X' = X\left(dX + ey - \frac{dc}{e}\right), \qquad y' = X - \frac{c}{e}. \tag{15}$$

Let $h = h(X, y)$ be an irreducible Darboux polynomial of system (17) with non-zero cofactor. Then it satisfies

$$X\left(dX + ey - \frac{dc}{e}\right)\frac{\partial h}{\partial X} + \left(X - \frac{c}{e}\right)\frac{\partial h}{\partial y} = (\alpha_0 + \alpha_1 X + \alpha_2 y)h. \tag{16}$$

We denote by $\bar{h} = \bar{h}(y)$ the restriction of $h$ to $X = 0$. Then $\bar{h} \neq 0$ and satisfies

$$-\frac{c}{e}\frac{d\bar{h}}{dy} = (\alpha_0 + \alpha_2 y)\bar{h} \quad \text{that is} \quad \bar{h} = K\exp\left(\frac{-ey(2\alpha_0 + \alpha_2 y)}{2c}\right), \quad K \in \mathbb{C} \setminus \{0\}.$$

Since $\bar{h}$ must be a polynomial we have $\alpha_0 = \alpha_2 = 0$ and $\bar{h} = K$. Note that $\alpha_1 \neq 0$ since otherwise the cofactor of $h$ would be zero. Therefore $h = K + Xg$ for some $g \in \mathbb{C}[X, y]$. Then introducing $h$ in (16) with $\alpha_0 = \alpha_2 = 0$ and removing the common factor $X$ we obtain

$$\left(dX + ey - \frac{dc}{e}\right)\left(g + X\frac{\partial g}{\partial X}\right) + \left(X - \frac{c}{e}\right)\frac{\partial g}{\partial y} = \alpha_1(K + Xg).$$

Let $\bar{g} = \bar{g}(y)$ be the restriction of $g$ to $X = 0$. Then it satisfies

$$-\frac{c}{e}\frac{d\bar{g}}{dy} + \left(ey - \frac{dc}{e}\right)\bar{g} = \alpha_1 K,$$

that is

$$\bar{g} = K_1\exp\left(-dy + \frac{e^2y^2}{2c}\right) + \frac{\alpha_1 K}{\sqrt{c}}\sqrt{\frac{\pi}{2}}\exp\left(\frac{cd^2}{2e^2} - dy + \frac{e^2y^2}{2c}\right)\operatorname{erf}\left(\frac{cd - e^2y}{\sqrt{2c}e}\right),$$

where $\operatorname{erf}(z)$ is the integral of the Gaussian distribution given by the entire function $\sqrt{\pi}\operatorname{erf}(z) = 2\int_0^z e^{-t^2}\,dt$, for more details see [1]. Since $\bar{g}$ must be a polynomial we must have $\alpha_1 K = 0$, a contradiction.

*Case* 2: $\gamma = -e/d$. In this case system (3) satisfying (S2) has $h_1 = x + ey/d$ as the unique irreducible Darboux polynomial of degree one with non-zero cofactor. We will see that this is in fact the only irreducible Darboux polynomial with non-zero cofactor. We introduce the variables $(X, y)$ with $X = h_1$. In these variables system (3) satisfying (S2) becomes

$$X' = X\left(dX - ey + \frac{dc}{e}\right), \qquad y' = X - \frac{e}{d}y. \tag{17}$$

Let $h = h(X, y)$ be an irreducible Darboux polynomial of system (17) with non-zero cofactor. Then it satisfies

$$X\left(dX - ey + \frac{dc}{e}\right)\frac{\partial h}{\partial X} + \left(X - \frac{e}{d}y\right)\frac{\partial h}{\partial y} = (\alpha_0 + \alpha_1 X + \alpha_2 y)h. \tag{18}$$

We denote by $\bar{h} = \bar{h}(y)$ the restriction of $h$ to $X = 0$. Then $\bar{h} \neq 0$ and satisfies

$$-\frac{e}{d}y\frac{d\bar{h}}{dy} = (\alpha_0 + \alpha_2 y)\bar{h} \quad \text{that is} \quad \bar{h} = Ky^{-\alpha_0 d/e}\exp(-\alpha_2 dy/e), \quad K \in \mathbb{C} \setminus \{0\}.$$

Since $\bar{h}$ must be a polynomial we have $\alpha_2 = 0$ and $\alpha_0 = -le/d$ for some non-negative integer $l$.

Now we write $h$ in sum of its homogeneous parts as $h = \sum_{j=0}^n h_j(X, y)$ where each $h_j$ is a homogeneous polynomial of degree $j$. We note that $h_n \neq 0$ and $n > 0$. Computing the terms of degree $n+1$ in (18) we have that

$$X(dX - ey)\frac{\partial h_n}{\partial X} = \alpha_1 X h_n \quad \text{that is} \quad h_n = K(y)(dX - ey)^{\alpha_1/d},$$

where $K(y)$ is a function in $y$ which is not zero. Since $h_n$ must be a polynomial we get $\alpha_1/d = m$ for some non-negative integer $m \in \mathbb{N}$ and $K(y) = c_n y^{n-m}$ with $c_n \in \mathbb{C} \setminus \{0\}$ and $0 \leqslant m \leqslant n$. Therefore

$$h_n = c_n(dX - ey)^m y^{n-m}, \quad c_n \in \mathbb{C} \setminus \{0\}.$$

Now if we compute the terms of degree $n$ in (18) we get

$$X(dX - ey)\frac{\partial h_{n-1}}{\partial X} + \frac{dc}{e}X\frac{\partial h_n}{\partial X} + \left(X - \frac{e}{d}y\right)\frac{\partial h_n}{\partial y} = dmXh_{n-1} - \frac{le}{d}h_n.$$

Solving it we obtain

$$\begin{aligned} h_{n-1} = {} & c_{n-1}y^{n-m-1}(dX - ey)^m \\ & - \frac{c_n}{de}y^{-m+n-1}(dX - ey)^m\left(-\frac{cd^2my}{dX - ey} + e(l - m)\log(dX - ey) - e(l - n)\log(dX)\right). \end{aligned}$$

Since $h_{n-1}$ must be a polynomial and $e \neq 0$ we have that $l = m = n$ with $n \geqslant 1$. Then

$$h_n = c_n(dX - ey)^n \quad \text{and} \quad h_{n-1} = \frac{c_n cdn}{e}(dX - ey)^{n-1}.$$

Now if we compute the terms of degree $n-1$ in (18) we have

$$X(dX - ey)\frac{\partial h_{n-2}}{\partial X} + \frac{dc}{e}X\frac{\partial h_{n-1}}{\partial X} + \left(X - \frac{e}{d}y\right)\frac{\partial h_{n-1}}{\partial y} = dnXh_{n-2} - \frac{ne}{d}h_{n-1}.$$

Solving it we get

$$h_{n-2} = \frac{c_n cn}{2e^2y^2}(dX - ey)^n\left(\frac{y(2deX - 2e^2y + cd^2(n-1)y)}{(dX - ey)^2} + 2\log\frac{dX - ey}{dX}\right).$$

Since $h_{n-2}$ must be a polynomial we must have $c_n cn = 0$, a contradiction. This concludes the proof of the lemma. □

**Lemma 10.** *System* (3) *satisfying* (S2) *with $de \neq 0$ and $b = cd/e + \gamma$ with $\gamma \notin \{0, -e/d\}$ has no Darboux polynomials with non-zero cofactor.*

**Proof.** We write $b = dc/e + \gamma$ and system (3) satisfying (S2) becomes

$$x' = \left(\frac{dc}{e} + \gamma\right)x + cy + dx^2 + exy, \qquad y' = x. \tag{19}$$

Note that $h$ satisfies

$$\left(\left(\frac{dc}{e} + \gamma\right)x + cy + dx^2 + exy\right)\frac{\partial h}{\partial x} + x\frac{\partial h}{\partial y} = (\alpha_0 + \alpha_1 x + \alpha_2 y)h. \tag{20}$$

We write $h$ in sum of its homogeneous parts as $h = \sum_{j=0}^n h_j(x, y)$ where each $h_j$ is a homogeneous polynomial of degree $j$ and $h_n \neq 0$ with $n > 0$. Computing the terms of degree $n+1$ in (20) we get

$$\left(dx^2 + exy\right)\frac{\partial h_n}{\partial x} = (\alpha_1 x + \alpha_2 y)h_n \quad \text{that is} \quad h_n = K(y)x^{\alpha_2/e}(dx + ey)^{\alpha_1/d - \alpha_2/e}$$

which yields $\alpha_2 = me$ for some non-negative integer $m$ and $\alpha_1/d = l + m$ for some non-negative integer $l$. Then $h_n = c_n y^{n-l-m}x^m(dx + ey)^l$ with $c_n \in \mathbb{C} \setminus \{0\}$.

Now computing the terms of degree $n$ in (20) we get that

$$\left(dx^2 + exy\right)\frac{\partial h_{n-1}}{\partial x} + \left[\left(\gamma + \frac{cd}{e}\right)x + cy\right]\frac{\partial h_n}{\partial x} + x\frac{\partial h_n}{\partial y} = \big(d(l+m)x + emy\big)h_{n-1} + \alpha_0 h_n,$$

which yields

$$\begin{aligned} h_{n-1} = {} & c_{n-1}x^m(dx + ey)^l y^{n-m-l-1} - \frac{c_n}{e}x^m(dx + ey)^l y^{n-m-l-1}\left[-\frac{c_m y}{x}\right. \\ & - \frac{e(e + d\gamma)ly}{d(dx + ey)} + \frac{1}{e}(-\alpha_0 e + cdl + e\gamma m)\log x \\ & \left. + \frac{1}{de}\big(\alpha_0 de - cd^2 l - e\big(d\gamma m + e(l + m - n)\big)\big)\log(dx + ey)\right]. \end{aligned}$$

Since $h_{n-1}$ must be a homogeneous polynomial of degree $n-1$ we must have

$$l = n - m, \qquad \alpha_0 = m\gamma + (n - m)\frac{cd}{e} \quad \text{and} \quad c_{n-1} = 0,$$

which yields

$$h_n = c_n x^m (dx + ey)^{n-m},$$

and

$$h_{n-1} = \frac{c_n}{de}x^{m-1}(dx + ey)^{n-m-1}\big(-e(e + d\gamma)(m - n)x + cdm(dx + ey)\big).$$

Now computing the terms of degree $n-1$ in (20) we obtain

$$\left(dx^2+exy\right)\frac{\partial h_{n-2}}{\partial x}+\left[\left(\gamma+\frac{cd}{e}\right)x+cy\right]\frac{\partial h_{n-1}}{\partial x}+x\frac{\partial h_{n-1}}{\partial y}=(dnx+emy)h_{n-2}+\left(m\gamma+(n-m)\frac{cd}{e}\right)h_{n-1}.$$

Solving it we get

$$\begin{aligned}h_{n-2}&=c_{n-2}x^m(dx+ey)^{n-m}y^{-2}\\&\quad+\frac{c_n}{2de^2y^2}x^m(dx+ey)^{n-m}\left(\frac{2cm(d\gamma(n-m-1)+e(n-m))y}{x}\right.\\&\quad+\frac{c^2d(m-1)my^2}{x^2}+\frac{e^2(e+d\gamma)^2(m-n)(1+m-n)y^2}{d(dx+ey)^2}\\&\quad+\frac{2cd(e+d\gamma)(m-1)(m-n)y}{dx+ey}+\frac{2cd(d\gamma(n-2m)+e(n-m))\log x}{e}\\&\quad\left.+\frac{2cd((e+2d\gamma)m-(e+d\gamma)n)\log(dx+ey)}{e}\right).\end{aligned}$$

Since $h_{n-2}$ must be a homogeneous polynomial of degree $n-2$, $c_n\gamma(d\gamma+e)\neq 0$ we get that either $m=n=0$, which is not possible, or $m=1$ and $n=(e+2d\gamma)/(e+d\gamma)$ (in the cases in which $(e+2d\gamma)/(e+d\gamma)\geqslant 1$ is an integer). In this last case we obtain

$$h_n=c_nx(dx+ey)^{d\gamma/(e+d\gamma)},\qquad h_{n-1}=\frac{c_n}{e}(dx+ey)^{-e/(e+d\gamma)}(cdx+e\gamma x+cey),$$

and

$$h_{n-2}=-\frac{e\gamma c_n}{2d}x(dx+ey)^{-1-e/(e+d\gamma)},$$

where we are also assuming that $-e/(e+d\gamma)>1$ is an integer. Now computing the terms of degree $n-2$ in (20) we obtain

$$\left(dx^2+exy\right)\frac{\partial h_{n-3}}{\partial x}+\left[\left(\gamma+\frac{cd}{e}\right)x+cy\right]\frac{\partial h_{n-2}}{\partial x}+x\frac{\partial h_{n-2}}{\partial y}=(dnx+emy)h_{n-3}+\left(m\gamma+(n-m)\frac{cd}{e}\right)h_{n-2}.$$

Solving it we get

$$\begin{aligned}h_{n-3}&=\frac{c_n\gamma}{6d^2e^3y^3}(dx+ey)^{-(3e+2d\gamma)/(e+d\gamma)}\Big[ey\big(e^3(2e+d\gamma)xy^2\\&\quad-3cd(dx+ey)\big(4d^2x^2+6dexy+e^2y^2\big)\big)-12cd^2x(dx+ey)^3\log\frac{x}{dx+ey}\Big].\end{aligned}$$

Since $h_{n-3}$ must be a homogeneous polynomial of degree $n-3$, we must have $2c_n\gamma c=0$, a contradiction. This concludes the proof of the theorem in this case. □

## 5. Exponential factors of system (3) satisfying (S2)

Let $E=\exp(g/h)$ be an exponential factor of system (3) satisfying (S2) with $b^2+d^2\neq 0$ and $b^2+e^2\neq 0$, having the cofactor $L=\beta_0+\beta_1x+\beta_2y$ where $\beta_i\in\mathbb{C}$ for $i=1,2,3$. In view of Propositions 3 and 4 $E$ must be of the form:

(E1) If either $d=0$ and $b\neq 0$, or $d\neq 0$ and $e=0$, or $de\neq 0$ and $b=cd/e+\gamma$ with $\gamma\notin\{0,-e/d\}$, then $E=\exp(g)$.
(E2) If $de\neq 0$ and $b=cd/e$, then $E=\exp(g/(x+c/e)^n)$ with $n\in\mathbb{N}\cup\{0\}$.
(E3) If $de\neq 0$ and $b=cd/e-e/d$, then $E=\exp(g/(x+ey/d)^n)$ with $n\in\mathbb{N}\cup\{0\}$.

Proposition 5 will follow from the following two lemmas.

**Lemma 11.** *System* (3) *satisfying* (S2) *with $d=0$ and $b\neq 0$ has the exponential factors* $\exp(y)$ *and* $\exp(-x/b+y+ey^2/(2b))$ *with cofactors $x$ and $-cy/b$, respectively. Furthermore, if either $d\neq 0$ and $e=0$, or $de\neq 0$ and $b=cd/e+\gamma$ with $\gamma\notin\{0,-e/d\}$, then* $\exp(y)$ *is the unique exponential factor with cofactor $x$.*

**Proof.** From (E1) we have that any exponential factor must be of the form $E=\exp(g)$ with $g\in\mathbb{C}[x,y]\setminus\mathbb{C}$ and cofactor $L=\beta_0+\beta_1x+\beta_2y$. Now we define

$$G=E\exp(-\beta_1y).$$

We claim that $G = \exp(T)$ with $T \in \mathbb{C}[x, y]$ is an exponential factor of system (3) satisfying (S2) with cofactor

$$L_1 = \beta_2 y, \quad \beta_2 \in \mathbb{C}.$$

Now we prove the claim. We have that

$$\begin{aligned} P(x, y)\frac{\partial G}{\partial x} + x\frac{\partial G}{\partial y} &= P(x, y)\frac{\partial E}{\partial x}\exp(-\beta_1 y) + x\frac{\partial E}{\partial y}\exp(-\beta_1 y) - \beta_1 x E\exp(-\beta_1 y) \\ &= (L - \beta_1 x)E\exp(-\beta_1 y) = (\beta_0 + \beta_2 y)G. \end{aligned}$$

Therefore, since $G$ is an exponential factor of system (3) satisfying (S2) with cofactor $\beta_0 + \beta_2 y$ we obtain that

$$P(x, y)\frac{\partial T}{\partial x} + x\frac{\partial T}{\partial y} = \beta_0 + \beta_2 y. \tag{21}$$

Evaluating (21) at $x = y = 0$ since system (3) satisfying (S2) has a unique fixed point at the origin, we get that $\beta_0 = 0$, and consequently $L_1 = \beta_2 y$. This proves the claim. We now consider two different cases.

*Case* 1: $d = 0$ *and* $b \neq 0$. Then system (3) satisfying (S2) becomes

$$x' = bx + cy + exy, \qquad y' = x,$$

with $ce \neq 0$. Let $T = T(x, y)$ be a polynomial satisfying (21), then

$$(bx + cy + exy)\frac{\partial T}{\partial x} + x\frac{\partial T}{\partial y} = \beta_2 y. \tag{22}$$

We write $T$ as a polynomial in the variable $y$ as $T = \sum_{j=0}^{n} T_j(x)y^j$. Assume $n \geqslant 3$. Computing the coefficient of $y^{n+1}$ in (22) we get

$$(c + ex)\frac{dT_n}{dx} = 0 \quad \text{that is} \quad T_n = c_n \in \mathbb{C}.$$

Computing the coefficient of $y^n$ in (22) we get

$$(c + ex)\frac{dT_{n-1}}{dx} = 0 \quad \text{that is} \quad T_{n-1} = c_{n-1} \in \mathbb{C}.$$

Furthermore, computing the coefficient of $y^{n-1}$ in (22) and since $n \geqslant 3$ we get

$$(c + ex)\frac{dT_{n-2}}{dx} + nc_n x = 0,$$

that is

$$T_{n-2} = c_{n-2} - nc_n\left(\frac{x}{e} - \frac{c}{e^2}\log(c + ex)\right), \quad c_{n-2} \in \mathbb{C}.$$

Since $T_{n-2}$ must be a polynomial we get that $nc_n c = 0$. Since $nce \neq 0$ we must have $c_n = 0$ and then $T_n = 0$. Therefore, we have proved that

$$T = T_0(x) + c_1 y + c_2 y^2.$$

From (22) we get

$$T_0(x) = c_0 + \frac{1}{(b + ey)^2}\bigl(-x(c_1 + 2c_2 y)(b + ey) + y\bigl(b\beta_2 + cc_1 + (2cc_2 + \beta_2 e)y\bigr)\log(bx + cy + exy)\bigr).$$

Since $T_0(x)$ must be a polynomial, we obtain

$$c_2 = \frac{ec_1}{2b}, \qquad \beta_2 = -\frac{cc_1}{b} \quad \text{and} \quad T_0(x) = c_0 - \frac{c_1}{b}x.$$

Therefore

$$T(x) = c_0 - \frac{c_1}{b}x + c_1 y + \frac{ec_1}{2b}y^2, \qquad L = -\frac{cc_1}{b}y, \quad c_0, c_1 \in \mathbb{C},$$

which proves the lemma in this case.

*Case* 2: *$d \neq 0$ and $e = 0$, or $de \neq 0$ and $b = cd/e + \gamma$ with $\gamma \notin \{0, -e/d\}$.* Therefore system (3) satisfying (S2) can be written as

$$x' = bx + cy + dx^2 + exy, \qquad y' = x. \tag{23}$$

Let $T = T(x, y)$ be a polynomial satisfying (21). We write it as a polynomial in the variable $x$ as $T(x, y) = \sum_{j=0}^{n} T_j(y)x^j$. Then computing the coefficient of $x^{n+1}$ in (21) with $n \geqslant 1$ we get

$$dnT_n + \frac{dT_n}{dy} = 0 \quad \text{that is} \quad T_n = c_n \exp(-dny), \quad c_n \in \mathbb{C}.$$

Since $T_n$ must be a polynomial we have that $c_n = 0$, and then $T_n = 0$ for $n \geqslant 1$. Therefore $T = T_0(y)$, and in view of (21) it satisfies

$$x\frac{dT_0}{dy} = \beta_2 y,$$

which yields $\beta_2 = 0$ and $T_0 \in \mathbb{C}$. Then the unique exponential factor in this case is $\exp(y)$. This concludes the proof of the lemma. □

**Lemma 12.** *The unique exponential factor for system* (3) *satisfying* (S2) *with $de \neq 0$, and $b = cd/e + \gamma$ with $\gamma \in \{0, -e/d\}$ is* $\exp(y)$ *with cofactor $x$.*

**Proof.** We consider two different cases.

*Case* 1: $\gamma = 0$. Under the assumptions of Lemma 12 and introducing the change of variables $(X, Y) = (x + c/e, y)$ we obtain that system (3) satisfying (S2) becomes (15). From (E2) we have $E = \exp(g/X^n)$ with $n \in \mathbb{N} \cup \{0\}$. It satisfies the equation

$$X\left(dX + eY - \frac{dc}{e}\right)\frac{\partial g}{\partial X} + \left(X - \frac{c}{e}\right)\frac{\partial g}{\partial Y} - n\left(dX + eY - \frac{dc}{e}\right)g = (\beta_0 + \beta_1 X + \beta_2 Y)X^n. \tag{24}$$

We consider two different cases.

*Case* 1.1: $n > 0$. In this case evaluating (24) on $X = 0$ and denoting by $\bar{g} = \bar{g}(Y) = g(0, Y)$ we get that

$$-\frac{c}{e}\frac{d\bar{g}}{dY} - n\left(eY - \frac{dc}{e}\right)\bar{g} = 0 \quad \text{that is} \quad \bar{g} = K \exp\left(dny - \frac{e^2ny^2}{2c}\right), \quad K \in \mathbb{C}.$$

Since $\bar{g}$ must be a polynomial we conclude that $K = 0$ and then $\bar{g} = 0$, a contradiction with the fact that $g$ is coprime with $X^n$. Therefore this case is not possible.

*Case* 1.2: $n = 0$. In this case we can write system (3) satisfying (S2) as in (23) with $b = cd/e$. Then proceeding as in the proof of Case 2 in Lemma 11 we get that the unique exponential factor is $\exp(y)$ with cofactor $x$.

*Case* 2: $\gamma = -e/d$. Under the assumptions of Lemma 12 and introducing the change of variables $(X, Y) = (x + ey/d, y)$ we obtain that system (3) satisfying (S2) becomes (17). In view of (E3) we have $E = \exp(g/X^n)$ with $n \in \mathbb{N} \cup \{0\}$. It satisfies the equation

$$X\left(dX - eY + \frac{dc}{e}\right)\frac{\partial g}{\partial X} + \left(X - \frac{e}{d}Y\right)\frac{\partial g}{\partial Y} - n\left(dX - eY + \frac{dc}{e}\right)g = (\beta_0 + \beta_1 X + \beta_2 Y)X^n. \tag{25}$$

We consider two different cases.

*Case* 2.1: $n > 0$. In this case evaluating (25) on $X = 0$ and denoting by $\bar{g} = \bar{g}(Y) = g(0, Y)$ we get

$$-\frac{e}{d}Y\frac{d\bar{g}}{dY} - n\left(-eY + \frac{dc}{e}\right)\bar{g} = 0 \quad \text{that is} \quad \bar{g} = KY^{-cd^2n/e^2} \exp(dnY), \quad K \in \mathbb{C}.$$

Since $\bar{g}$ must be a polynomial we conclude that $K = 0$, and then $\bar{g} = 0$, a contradiction with the fact that $g$ is coprime with $X^n$. Therefore this case is not possible.

*Case* 2.2: $n = 0$. In this case we can write system (3) satisfying (S2) as in (23) with $b = cd/e - e/d$. Then proceeding as in the proof of Case 2 in Lemma 11 we get that the unique exponential factor is $\exp(y)$ with cofactor $x$. This concludes the proof of the lemma. □

## 6. Proof of Theorem 2

We have proved that if $d = b = 0$ or $b = e = 0$, then we have the first integrals of the statements of Theorem 2. Now it remains to show that when $b^2 + d^2 \neq 0$ or $b^2 + e^2 \neq 0$ there are no Liouvillian first integrals. By Theorem 7 in order that system (3) satisfying (S2) has a Liouvillian first integral, it must have an integrating factor given by a Darboux function (see (9)). We consider four different cases.

*Case* 1: $d = 0$ *and* $b \neq 0$. Then from Propositions 3 and 5 and Theorem 6(b) system (3) satisfying (S2) has an integrating factor which is a Darboux function if and only if

$$\mu_1 x - \frac{\mu_2 c}{b} y = -b - ey,$$

which is not possible since $b \neq 0$.

*Case* 2: $d \neq 0$, $e = 0$ *and* $b \neq 0$, *or* $de \neq 0$ *and* $b = cd/e + \gamma$ *with* $\gamma \notin \{0, -e/d\}$. Therefore from Propositions 3 and 5 and Theorem 6(b) system (3) satisfying (S2) has an integrating factor which is a Darboux function if and only if

$$\mu_1 x = -b - 2dx - ey,$$

which is not possible since $e^2 + b^2 \neq 0$.

*Case* 3: $de \neq 0$ *and* $b = cd/e$. In this case from Propositions 3 and 5 and Theorem 6(b) system (3) satisfying (S2) has an integrating factor which is a Darboux function if and only if

$$\mu_1 x + \mu_2 (dx + ey) = -\frac{cd}{e} - 2dx - ey,$$

which is impossible since $cd/e \neq 0$.

*Case* 4: $de \neq 0$ *and* $b = cd/e - e/d$. From Propositions 3 and 5 and Theorem 6(b) system (3) satisfying (S2) has an integrating factor which is a Darboux function if and only if

$$\mu_1 x + \frac{\mu_2 d}{e}(c + ex) = -\frac{cd}{e} + \frac{e}{d} - 2dx - ey,$$

which is impossible since $e \neq 0$. This concludes the proof of the theorem.

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